1. Linearised gravity eqn for $\bar{h}_{ij} = h_{ij} - \frac{1}{2}\eta_{ij} h$

$$\Box \bar{h}_{ij} = -\frac{16\pi G}{c^4} T_{ij} \quad (1) \qquad Tr\, \bar{h}_{ij} = -h.$$

This was derived in two ways yesterday — first by calculating the Ricci tensor, Einstein tensor to order 'h', second by using the field theoretic approach to gravity (linearised) via the Fierz-Pauli Lagrangian density involving the rank 2 symmetric tensor field h_{ij}.

We have also used the gauge condition

$$\partial^i \bar{h}_{ij} = 0 \quad (2)$$

known as the de Donder or the Lorentz gauge.

Note that (2) gives four conditions. Therefore out of the ~~six~~ ten independent $\bar{h}_{ij}$ only six remain. However, all of the six do not represent gravitational wave degrees of freedom. Ultimately, there are only two true radiative degrees of freedom. We will see this more explicitly at the end of the lecture.

What is the formal solution of Eqn (1)? We may write (for $T_{ij}=0$)

$$\bar{h}_{ij} = \int d^3k\, A_{ij}(\underline{k})\, e^{i(k_\ell x^\ell)}.$$

This is the usual Fourier way of writing a solution.

Note that in Fourier space the gauge condition becomes

$$k^i A_{ij} = 0$$

Let us now assume $k^i = (1, n^\alpha)$ where $n^\alpha n_\alpha = 1$.

Thus n^α represents the direction of propagation of the wave.

What happens to the condition $k^i A_{ij} = 0$ for $j = 0, \beta$?

$j=0$ $\quad k^i A_{i0} = k^0 A_{00} + k^\alpha A_{\alpha 0} = A_{00} + n^\alpha A_{\alpha 0} = 0$

This translates into $\quad A_{00} = -n^\alpha A_{\alpha 0} = -n^\alpha A_{\alpha 0}$

$j = \beta$ $\quad k^i A_{i\beta} = k^0 A_{0\beta} + k^\alpha A_{\alpha\beta} = A_{0\beta} + n^\alpha A_{\alpha\beta} = 0$

Hence $\quad n^\beta A_{0\beta} = -n^\beta n^\alpha A_{\alpha\beta}$

Thus $\quad A_{00} = n^\alpha n^\beta A_{\alpha\beta}.$

Let us now consider the 4×4 matrix

$$A_{ij} = \begin{bmatrix} A_{00} & A_{01} & A_{02} & A_{03} \\ A_{01} & A_{11} & A_{12} & A_{13} \\ A_{02} & A_{12} & A_{22} & A_{23} \\ A_{03} & A_{13} & A_{23} & A_{33} \end{bmatrix}$$

How does this matrix look like when we impose $A_{00} = A_{\alpha\beta} n^\alpha n^\beta$ and $A_{0\beta} = -n^\alpha A_{\alpha\beta}$

Let us assume $n^\alpha = (0,0,1) \rightarrow$ wave propagating along the z direction.

For this case, $A_{00} = A_{33}$ and. $A_{03} = -A_{33} = -A_{00}$

$A_{01} = -A_{31}$, $A_{02} = -A_{32}$,

Hence

$$A_{ij} = \begin{bmatrix} A_{00} & -A_{13} & -A_{23} & -A_{00} \\ -A_{13} & A_{11} & A_{12} & A_{13} \\ -A_{23} & A_{12} & A_{22} & A_{23} \\ -A_{00} & A_{13} & A_{23} & A_{00} \end{bmatrix}$$

Six independent Components.

We now introduce a new gauge with the additional constraints

$$h_{00} = h_{0\alpha} = 0 \; , \; \underbrace{h = 0}_{\bar{h}_{ij} = \bar{h}_{ij}'} \qquad \text{(we are in vacuum so } \Box \bar{h}_{ij} = \Box h_{ij} \text{)}$$

These are four additional constraints. from $h_{00} = h_{0\alpha} = 0$ and 1 from $h=0$

We had 6. Now with 4 more constraints we have 2 left. The condition $\partial^i \bar{h}_{ij} = 0 \Rightarrow \partial^\alpha h_{\alpha\beta} = 0$ (3 conditions)

Thus, we have in total 8 constraints for the 10 h_{ij}

This is the TRANSVERSE, TRACELESS GAUGE. (TT gauge).

The old condition $h'_{ij} = h_{ij} - \partial_i \xi_j - \partial_j \xi_i$ leads to

$$h' = h - 2\partial_i \xi^i$$

Thus $$h'_{ij} - \tfrac{1}{2}\eta_{ij} h' = h_{ij} - \partial_i \xi_j - \partial_j \xi_i - \tfrac{1}{2}\eta_{ij}(h - 2\partial_k \xi^k)$$

or, $$\bar{h}'_{ij} = \bar{h}_{ij} - \partial_i \xi_j - \partial_j \xi_i + \eta_{ij}\partial_k \xi^k. \quad (3)$$

$$\Rightarrow \quad \partial^i \bar{h}'_{ij} = \partial^i \bar{h}_{ij} - \Box \xi_j - \cancel{\partial^i \partial_j \xi_i} + \cancel{\eta^i_j \partial_k} \xi^k.$$

or, $$\partial^i \bar{h}'_{ij} = \partial^i \bar{h}_{ij} + \Box \xi_j$$

Thus if $\partial^i \bar{h}_{ij} \neq 0$ then by choosing a ξ_j s.t.

$$\Box \xi_j = \partial^i \bar{h}_{ij}$$ one can find a $\bar{h}_{ij}'$ which

satisfies the gauge condition.

On the other hand, if we want $\bar{h}_{ij}'$ to satisfy the gauge condition $\partial^i \bar{h}'_{ij} = 0$ when $\bar{h}_{ij}$ does satisfy it, we need $\Box \xi_j = 0$.

We assume this last case and take $\xi_i = B_i e^{-ik_\ell x^\ell}$.

Writing $\bar{h}_{ij} = A_{ij} e^{-ik_\ell x^\ell}$, $\bar{h}_{ij}' = A_{ij}' e^{-ik_\ell x^\ell}$

We can rewrite Eqn ③ as follows: ④

$$A_{ij}' = A_{ij} + i k_i B_j + i k_j B_i + i \eta_{ij} k_k B^k .$$

We will now impose the conditions $Tr\, A_{ij}' = 0$, $A_{00}' = 0$ and $A'_{0\alpha} = 0$ to set the TT gauge.

Thereafter, we will find an expression for $A'_{\alpha\beta}$ in terms of $A_{\alpha\beta}$.

Let us first calculate the trace of A_{ij}

$$Tr\, A = \eta^{ij} A_{ij} = - A_{00} + A^{\alpha}{}_{\alpha}$$

$$= - A_{\alpha\beta}\, n^{\alpha} n^{\beta} + A_{\alpha\beta}\, \delta^{\alpha\beta}$$

$$= \left(\delta^{\alpha\beta} - n^{\alpha} n^{\beta} \right) A_{\alpha\beta}$$

We define $P^{\alpha\beta} = \delta^{\alpha\beta} - n^{\alpha} n^{\beta}$

This is the projection operator. Note that $P^{\alpha\beta} P_{\beta\gamma} = P^{\alpha}{}_{\gamma}$ ($P^2 = P$, idempotent).

$$\therefore \quad Tr\, A = P^{\alpha\beta} A_{\alpha\beta} .$$

Now $\underline{Tr\, A' = 0} = Tr\, A + i k^i B_i + i k^i B_i - 4i\, k_k B^k$

$$= Tr\, A - 2i\, k^k B_k .$$

$$\therefore \quad i k^k B_k = \frac{1}{2} Tr\, A = \frac{1}{2} P^{\alpha\beta} A_{\alpha\beta} .$$

Hence,

$$A'_{ij} = A_{ij} + i k_i B_j + i k_j B_i - \eta_{ij} \frac{1}{2} P_{\alpha\beta} A^{\alpha\beta} . \qquad ⑤$$

What is $A'_{0\alpha} = 0$? And $A'_{00} = 0$.

$$0 = A_{0\alpha} + i k_0 B_\alpha + i k_\alpha B_0 \qquad (6)$$

$$0 = A_{00} + i k_0 B_0 + i k_0 B_0 + \tfrac{1}{2} P_{\alpha\beta} A^{\alpha\beta}.$$

$$0 = A_{00} + 2i k_0 B_0 + \tfrac{1}{2} P_{\alpha\beta} A^{\alpha\beta}. \qquad (7)$$

Using (7) we have with $k_0 = -1$.

$$2i B_0 = A_{\alpha\beta} n^\alpha n^\beta + \tfrac{1}{2} P_{\alpha\beta} A^{\alpha\beta}$$

$$i B_0 = \tfrac{1}{2} A_{\alpha\beta} n^\alpha n^\beta + \tfrac{1}{4} P_{\alpha\beta} A^{\alpha\beta}. \qquad (8)$$

use this $i B_0$ in (6). This gives.

$$+ i B_\alpha = A_{0\alpha} + i n_\alpha B_0$$

$$= -n^\beta A_{\alpha\beta} + n_\alpha \left[\tfrac{1}{2} A_{\gamma\beta} n^\gamma n^\beta + \tfrac{1}{4} P_{\gamma\beta} A^{\gamma\beta} \right]$$

$$i B_\alpha = -n^\beta A_{\alpha\beta} + \tfrac{1}{2} n_\alpha A_{\gamma\beta} n^\gamma n^\beta + \tfrac{1}{4} n_\alpha P_{\gamma\beta} A^{\gamma\beta} \qquad (9)$$

Eqns (8) and (9) give the gauge fns B_0, B_α in terms of $A_{\alpha\beta}$ and $P_{\alpha\beta}$.

Finally, we need to evaluate $A'_{\alpha\beta}$

$$A'_{\alpha\beta} = A_{\alpha\beta} + i k_\alpha B_\beta + i k_\beta B_\alpha - \delta_{\alpha\beta} \tfrac{1}{2} P_{\gamma\delta} A^{\gamma\delta}$$

We need to substitute for B_β, B_α and simplify.

$$A'_{\alpha\beta} = A_{\alpha\beta} + i n_\alpha B_\beta + i n_\beta B_\alpha - \delta_{\alpha\beta} \tfrac{1}{2} P_{\gamma\delta} A^{\gamma\delta}$$

We have $\quad i B_\beta = - n^\gamma A_{\gamma\beta} + \tfrac{1}{2} n_\beta A_{\gamma\delta} n^\gamma n^\delta + \tfrac{1}{4} n_\beta P_{\gamma\delta} A^{\gamma\delta}$

$$A'_{\alpha\beta} = A_{\alpha\beta} + n_\alpha \left[- n^\gamma A_{\gamma\beta} + \tfrac{1}{2} n_\beta A_{\gamma\delta} n^\gamma n^\delta + \tfrac{1}{4} n_\beta P_{\gamma\delta} A^{\gamma\delta} \right]$$
$$+ n_\beta \left[- n^\gamma A_{\gamma\alpha} + \tfrac{1}{2} n_\alpha A_{\gamma\delta} n^\gamma n^\delta + \tfrac{1}{4} n_\alpha P_{\gamma\delta} A^{\gamma\delta} \right]$$
$$- \tfrac{1}{2} \delta_{\alpha\beta} P_{\gamma\delta} A^{\gamma\delta}$$

$$= A_{\alpha\beta} - n_\alpha n^\gamma A_{\gamma\beta} + \tfrac{1}{2} n_\alpha n_\beta A_{\gamma\delta} n^\gamma n^\delta + \tfrac{1}{4} n_\alpha n_\beta P_{\gamma\delta} A^{\gamma\delta}$$
$$- n_\beta n^\gamma A_{\gamma\alpha} + \tfrac{1}{2} n_\alpha n_\beta A_{\gamma\delta} n^\gamma n^\delta + \tfrac{1}{4} n_\alpha n_\beta P_{\gamma\delta} A^{\gamma\delta}$$
$$- \tfrac{1}{2} \delta_{\alpha\beta} P_{\gamma\delta} A^{\gamma\delta}$$

$$= A_{\alpha\beta} - n_\alpha n^\gamma A_{\gamma\beta} - n_\beta n^\gamma A_{\gamma\alpha} \underline{- \tfrac{1}{2} \delta_{\alpha\beta} P_{\gamma\delta} A^{\gamma\delta}}.$$
$$+ n_\alpha n_\beta A_{\gamma\delta} n^\gamma n^\delta + \underline{\tfrac{1}{2} n_\alpha n_\beta P_{\gamma\delta} A^{\gamma\delta}}.$$

$$P^\gamma_\alpha = \delta^\gamma_\alpha - n^\gamma n_\alpha \rightarrow n^\gamma n_\alpha = \delta^\gamma_\alpha - P^\gamma_\alpha.$$

$$A'_{\alpha\beta} = \underline{A_{\alpha\beta} - n_\alpha n^\gamma A_{\gamma\beta} - n_\beta n^\gamma A_{\gamma\alpha} + n_\alpha n_\beta A_{\gamma\delta} n^\gamma n^\delta}$$
$$- \tfrac{1}{2} P_{\alpha\beta} P_{\gamma\delta} A^{\gamma\delta}$$

$$= P^\gamma_\alpha P^\delta_\beta A_{\gamma\delta} - \tfrac{1}{2} P_{\alpha\beta} P_{\gamma\delta} A^{\gamma\delta}$$

$$= \left(P^\gamma_\alpha P^\delta_\beta - \tfrac{1}{2} P_{\alpha\beta} P^{\gamma\delta} \right) A_{\gamma\delta}$$

$$\therefore \quad A'_{\alpha\beta} = \left(P^\gamma_\alpha P^\delta_\beta - \tfrac{1}{2} P_{\alpha\beta} P^{\gamma\delta} \right) A_{\gamma\delta}$$

Finally therefore, we have

$$i\, B_\alpha = - n^\beta A_{\alpha\beta} + \frac{1}{2} n_\alpha A_{\gamma\beta} n^\gamma n^\beta + \frac{1}{4} n_\alpha P_{\gamma\beta} A^{\gamma\beta} \qquad (7)$$

$$i\, B_0 = \frac{1}{2} A_{\alpha\beta} n^\alpha n^\beta + \frac{1}{4} P_{\alpha\beta} A^{\alpha\beta}$$

$$A'_{\alpha\beta} = \left(P^\gamma_\alpha P^\delta_\beta - \frac{1}{2} P_{\alpha\beta} P^{\gamma\delta} \right) A_{\gamma\delta} .$$

If $n^\alpha = (0, 0, 1)$ $\qquad P_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$i\, B_0 = \frac{1}{2} A_{33} + \frac{1}{4} P_{11} A^{11} + \frac{1}{4} P_{22} A^{22}$$

$$= \frac{1}{2} A_{00} + \frac{1}{4} \left(A_{11} + A_{22} \right)$$

$$i\, B_1 = - n^3 A_{13} + 0 + 0 = - A_{13} .$$

$$i\, B_2 = - n^3 A_{23} + 0 + 0 = - A_{23}$$

$$i\, B_3 = - n^3 A_{33} + \frac{1}{2} A_{33} + \frac{1}{4} \left(A_{11} + A_{22} \right)$$

$$= - \frac{1}{2} A_{33} + \frac{1}{4} \left(A_{11} + A_{22} \right) = - \frac{1}{2} A_{00} + \frac{1}{4} \left(A_{11} + A_{22} \right)$$

$$A'_{11} = \frac{1}{2} \left(A_{11} - A_{22} \right) \qquad A'_{22} = \frac{1}{2} \left(A_{22} - A_{11} \right)$$

Thus

$$A'_{ij} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}(A_{11} - A_{22}) & A_{12} & 0 \\ 0 & A_{12} & \frac{1}{2}(A_{22} - A_{11}) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

'TT gauge'

2. Go to slides for the scalar-vector-tensor decomposition of the perturbations

Demonstrates that only $h_{\alpha\beta}^{TT}$ satisfies a wave eqn. Other variables (gauge invariant) satisfy Laplace / Poisson eqns.

Thus $h_{\alpha\beta}^{TT}$ constitute the true radiative degrees of freedom of the <u>GW</u>.

Using $h_{\alpha\beta}^{TT}$ simplifies several expressions

A prime example is the Riemann tensor which becomes equal to

Component $R^{0}{}_{\alpha 0 \beta}$ extensively in the next

$\ddot{h}_{\alpha\beta}$. We shall use this

lecture.

The geodesic and geodesic deviation eqns also simplify and yield useful physical information on the propagation of gravitational waves.

Geodesic and deviation eqn in TT gauge.

$$\Gamma^i_{jk} = \frac{1}{2}\eta^{il}\left[h_{jl,k} + h_{kl,j} - h_{jk,l}\right]$$

$$h^{TT}_{ij} \Rightarrow h^{TT}_{00} = 0 \ , \ h^{TT}_{0\alpha} = 0$$

$$\Gamma^0_{jk} = \frac{1}{2}\eta^{00}\left[h^{TT}_{0j,k} + h^{TT}_{0k,j} - h^{TT}_{jk,0}\right]$$

$\therefore$ only $\Gamma^0_{\alpha\beta} = \frac{1}{2}\dot{h}^{TT}_{\alpha\beta}$ survives.

$$\Gamma^\alpha_{jk} = \frac{1}{2}\eta^{\alpha l}\left[h^{TT}_{lj,k} + h^{TT}_{lk,j} - h^{TT}_{jk,l}\right]$$

$$\Gamma^\alpha_{00} = 0 \ ; \ \Gamma^\alpha_{0\beta} = \frac{1}{2}\eta^{\alpha\alpha}\left[h^{TT}_{\alpha 0,\beta} + h^{TT}_{\alpha\beta,0} - h_{0\beta,\alpha}\right]$$

$$= \frac{1}{2}\dot{h}_{\alpha\beta,}\eta^{\alpha\alpha}$$

Similarly $\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}\eta^{\alpha\alpha}\left[h^{TT}_{\alpha\beta,\gamma} + h^{TT}_{\alpha\gamma,\beta} - h_{\beta\gamma,\alpha}\right]$

Since $\Gamma^\alpha_{00} = 0$, we have to lowest order in small parameters $\frac{d^2x^\alpha}{dt^2} + \Gamma^\alpha_{00} = 0 = \frac{d^2x^\alpha}{dt^2}$

Hence if $h^{TT}_{\alpha\beta}$ then x^α are not affected by it.

Therefore to linear order in h and h in the TT gauge x^α moves along with the wave and remains unaffected. The x^α behave like free coordinates $\rightarrow$ in 'free fall'

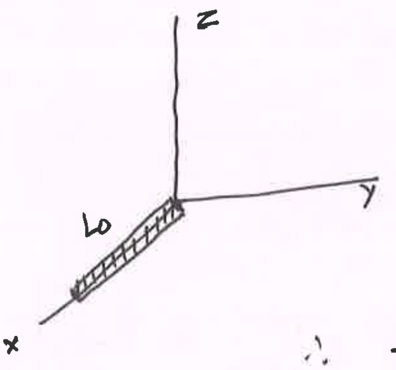
Note this is a result for linearised gravity.

However, proper distances are affected. How?

Take

$$ds^2 = -dt^2 + \left(1 + h_+(t,z)\right) dx^2 + \left(1 - h_+(t,z)\right) dy^2 + dz^2$$

Rod of length L_0 along 'x' in absence of wave.



What is the proper length in the presence of a wave?

$$L = \sqrt{1 + h_+(t,z)}\, L_0 = \left(1 + \tfrac{1}{2} h_+\right) L_0$$

$$\therefore \frac{\Delta L}{L_0} = \frac{1}{2} h_+$$

Strain

metric fn in GW metric.

Riemann tensor

$$R^t{}_{\alpha t \beta} = \partial_t \Gamma^t_{\alpha\beta} - \partial_\beta \Gamma^t_{\alpha t} + O(h^2)$$

$$= \partial_t \Gamma^t_{\alpha\beta} = \partial_t \left(\tfrac{1}{2} \dot{h}_{\alpha\beta}\right)$$

$$= \tfrac{1}{2} \ddot{h}_{\alpha\beta}$$

$u^i = (1,0,0,0) \rightarrow$ Deviation eqn

$$\frac{d^2 \eta^i}{dt^2} = -R^i{}_{tnt}\, \eta^n$$

$i = \alpha$ $\qquad \dfrac{d^2 \eta^\alpha}{dt^2} = -R^\alpha{}_{tnt}\, \eta^n = -R^\alpha{}_{t\beta t}\, \eta^\beta$

$\eta^\alpha = L^\alpha + \delta L^\alpha$ or $\eta_\alpha = L_\alpha + \delta L_\alpha$

$$\therefore \delta \ddot{L}^\alpha = -\eta^{\alpha\gamma} R_{\gamma t \beta t}\, \eta^\beta = -\eta^{\alpha\gamma} \eta_{tt} R^t{}_{\gamma t \beta}\, \eta^\beta$$

$$= \eta^{\alpha\gamma}\, \tfrac{1}{2}\, \ddot{h}_{\gamma\beta}\, \eta^\beta$$

$$\delta \ddot{L}^{\alpha} \simeq \frac{1}{2} \ddot{h}^{\alpha}{}_{\beta} L^{\beta} \quad \text{ignoring } h.\,\delta L$$

or, $$\delta L_{\alpha} = \frac{1}{2} h_{\alpha\beta} L^{\beta} \quad \text{Change in deviation.}$$

Change in deviation proportional to $h_{\alpha\beta}$ to linear order in 'h'

Let $n^{\alpha(1)}$ $n^{\beta(1)}$ be the arm-vectors of the interferometer.

$$\delta L_1 = n^{\alpha(1)} \delta L_{\alpha} = \frac{1}{2} h_{\alpha\beta} n^{\alpha(1)} n^{\beta(1)} L$$

$$\delta L_2 = n^{\alpha(2)} \delta L_{\alpha} = \frac{1}{2} h_{\alpha\beta} n^{\alpha(2)} n^{\beta(2)} L$$

$$\therefore \frac{\delta L_1 - \delta L_2}{L} = \frac{1}{2} h_{\alpha\beta} \underbrace{\left[n^{\alpha(1)} n^{\beta(1)} - n^{\alpha(2)} n^{\beta(2)} \right]}_{\text{'Detector tensor'}}$$

Write $$h_{\alpha\beta} = h_{+} e^{+}_{\alpha\beta} + h_{\times} e^{\times}_{\alpha\beta}$$

where $$e^{+}_{\alpha\beta} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad e^{\times}_{\alpha\beta} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \frac{\delta L_1 - \delta L_2}{L} = h_{+} \left(D_{\alpha\beta} e^{\alpha\beta +} \right) + h_{\times} \left(D_{\alpha\beta} e^{\alpha\beta \times} \right)$$

$$= h_{+} F_{+} + h_{\times} F_{\times}$$

$$F_{+} = D^{\alpha\beta} e^{+}_{\alpha\beta} \qquad F_{\times} = D^{\alpha\beta} e^{\times}_{\alpha\beta}$$

are the antenna pattern fns.

One can choose the detector frame basis as

Comprised of $e_{(x)}$, $e_{(y)}$, $e_{(z)}$

Wave frame basis $e_{(X)}$, $e_{(Y)}$, $e_{(Z)}$

$$\therefore \quad D^{\alpha\beta} = \left[e^{\alpha}_{(x)} e^{\beta}_{(x)} - e^{\alpha}_{(y)} e^{\beta}_{(y)} \right].$$

$$e^{+}_{\alpha\beta} = \left[e_{\alpha(X)} e_{\beta(X)} - e_{\alpha(Y)} e_{\beta(Y)} \right]$$

$$e^{\times}_{\alpha\beta} = \left[e_{\alpha(X)} e_{\beta(Y)} + e_{\alpha(Y)} e_{\beta(X)} \right].$$

Using these and the relation between the xyz and XYZ basis through the Euler angle matrix we can construct F_+ , $F_\times$ in maths.

These are the Gel'fand fns

Dhurandhar and Tinto 1986-87